

How Chiral Symmetry Breaking Affects the Spectrum of the Light-Heavy Mesons in the 't Hooft Model

L.Ya. Glozman,¹ V.K. Sazonov,^{1,*} M. Shifman,² and R.F. Wagenbrunn¹

¹*Institut für Physik, FB Theoretische Physik, Universität Graz, Universitätsplatz 5, A-8010 Graz, Austria*

²*William I. Fine Theoretical Physics Institute, University of Minnesota, Minneapolis, MN 55455, USA*

We find the spectrum and wave functions of the heavy-light mesons in $(1+1)$ -dimensional QCD in the 't Hooft limit, both in the rest frame, using the Coulomb (axial) gauge, and on the light cone. Our emphasis is on the effects of chiral symmetry breaking on the spectrum. While dynamical equations in both cases look different, the results for the spectrum are identical. The chiral symmetry breaking is clearly seen from the gap and Bethe–Salpeter equations in the laboratory frame. At the same time, while vacuum is trivial on the light cone (no chiral condensate), the effects of the spontaneous breaking of the chiral symmetry manifest themselves in the same way, as it follows from the coincidence of the spectra obtained from the laboratory-frame Bethe–Salpeter equation on the one hand, and the light-cone 't Hooft-type equation on the other.

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I. INTRODUCTION

In this paper we discuss the relation between the chiral symmetry breaking in the two-dimensional 't Hooft model [1] and the heavy-light meson mass spectrum.

The action of the version of the 't Hooft model we will consider is

$$S = \int d^2x \left[-\frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a + \sum_{f=1,2} \bar{\psi}_f (i \not{D} - m_f) \psi_f \right] \quad (1)$$

where $G_{\mu\nu}^a$ is the gluon field strength tensor, the index a runs from 1 to $N^2 - 1$, and N is the number of colors,

$$N \rightarrow \infty.$$

The subscript f marks quarks of different flavors. The quarks are assumed to belong to the fundamental representation of the gauge group $SU(N)$. Moreover, in our consideration we will assume that $m_2 \rightarrow \infty$, so that the second quark will play the role of a static force center, while $m_1 \rightarrow 0$ so that the first quark is massless. The theory then possesses two $U(1)$ symmetries, generated by the vector and axial currents, $\bar{\psi}_1 \gamma^\mu \psi_1$ and $\bar{\psi}_1 \gamma^\mu \gamma^5 \psi_1$, respectively. The axial symmetry is spontaneously broken (see below).

The coupling constant g has dimension of mass, and in the large- N limit scales as

$$\lambda \equiv \frac{g^2 N}{4\pi} = \text{const}. \quad (2)$$

The constant λ is referred to as the 't Hooft coupling.

The very fact of confinement is obvious in this model since in two dimensions the Coulomb potential generated

by the static color source (i.e. the infinitely heavy quark at the origin) grows linearly with separation. The model was solved in the light-cone formalism by 't Hooft [1] and further developed along the same lines in Refs. [2, 3]. The spectrum of the light-light mesons and the light-cone wave functions were obtained from the 't Hooft equation, an integral equation, supplemented by certain boundary conditions, well studied in the literature (for a review see e.g. [4]).

In the light-cone formalism one chooses the light-cone gauge condition,

$$A_- = 0.$$

The light-cone time derivative of A_+ does not appear in $G_{\mu\nu}$; hence, A_+ is a non-dynamical degree of freedom which can be eliminated through the equations of motion. In the large- N limit the only surviving diagrams are ladders and rainbows. The 't Hooft equation for the bound state built from the quark of the first flavor and anti-quark of the second flavor has the form

$$\left(\frac{m_1^2}{x} + \frac{m_2^2}{1-x} - M^2 \right) \phi(x) = 2\lambda \int_0^1 \frac{\phi(y) - \phi(x)}{(x-y)^2} dy, \quad (3)$$

where x is the first quark's share of the total (light-cone) momentum of the composite meson with mass M . If we deal with massless (anti)quarks in the equation above ($m_1 = m_2 = 0$), Eq. (3) has a massless-meson solution ("pion" with $M = 0$) which is known exactly. The corresponding light-cone wave function is x -independent, $\phi(x) = \text{const}$. The existence of the massless pion implies [5], through the standard current algebra relations, a non-vanishing quark condensate [17] $\langle \bar{\psi}\psi \rangle$ proportional to $-N\sqrt{\lambda}$, see also [6–8]. The problem is that this chiral condensate is not seen directly in the light-cone consideration, a usual story with all light-cone analyses of the vacuum condensates. The chiral condensate on the light cone is buried somewhere in zero modes and boundary conditions.

*Also at the Department of Theoretical Physics, St. Petersburg University, Uljanovskaja 1, St. Petersburg, Petrodvorez, 198504 Russia

Indeed, if one tries to extract the quark condensate directly from the light-cone quark Green's function given by 't Hooft, one obtains

$$\langle \bar{\psi}\psi \rangle \propto \lim_{x \rightarrow 0} \text{Tr} \{ S(x, 0) \} , \quad (4)$$

where $S(x, 0)$ is the massless quark Green's function describing the quark propagation from the point 0 to the point x . The right-hand side vanishes after taking trace, since this Green's function is linear in the γ matrices.

Our task is not only to reveal the chiral condensate (this had been already done by shifting slightly away from the light cone [7] or, from the solution of the gap equation in the laboratory frame [9]), but also to analyze its impact on the spectrum of bound mesons. In order to keep a closed-form integral equation à la 't Hooft as the spectral equation we have to focus on a system of an infinitely heavy anti-quark at rest at the origin and a dynamical quark of mass $m_1 \rightarrow 0$ bound by a linearly growing potential, i.e. the heavy-light quark system. The bound quark is ultra-relativistic, and dynamical details of its binding crucially depend on the chiral condensate (see below). At the same time, the system in question can be considered in the laboratory frame (as opposed to the light-cone consideration). The static infinitely heavy (anti)quark suppresses the so called Z graphs in much the same way as the transition to the light cone in the case of two massless (anti)quarks. The absence of the Z graphs is necessary to keep the spectral equation in the closed form. The above integral equation applies to the one-particle wave function in the momentum space. It can be readily obtained from the general analysis of [9] in the limit $m_2 \rightarrow \infty$ and $m_1 \rightarrow 0$. We will briefly review the derivation below.

Another aspect, to be addressed below, is the relation with the “original” light-cone spectral equation for the heavy-light system, which we will refer to as the 't Hooft-like equation. It was obtained [11, 12] from the general light-cone 't Hooft equation valid for arbitrary $m_{1,2}$ in the limit $m_2 \rightarrow \infty$ and $m_1 \rightarrow 0$. In fact, we deal with *two different* one particle equations. One of them is just a limiting case of the 't Hooft equation, and applies to the light-cone wave function, which depends on x ($0 \leq x \leq 1$). Within this approach the (massless quark) condensate vanishes. At the same time, our laboratory frame equation has the condensate built in. It is the spectral equation for $\phi(p)$ where p is the light-quark momentum in the laboratory frame. In deriving these two equations one uses two distinct limiting procedures. To obtain the 't Hooft-like equation one first tends the momentum to infinity, keeping the quark masses fixed, and then tends one of the quark masses to infinity. At the same time, when one works in the laboratory frame, one keeps the total momentum fixed and sends the quark mass to infinity from the very beginning. Generally speaking, these two limits need not be commutative.

Our analysis will demonstrate that the above two equations are, in fact, *isospectral*; i.e. the limiting procedures are interchangeable, with no obstructions.

Surprisingly, the laboratory frame equation for $\phi(p)$ *formally* becomes identical to the 't Hooft-like equation for $\varphi(\xi)$ (see Eq. (6)) upon substitution into the laboratory-frame equation a “wrong” solution for the chiral angle (i.e. a singular solution with no chiral symmetry breaking) and a rescaling of the overall energy scale. This curious coincidence has no obvious physical reason; at least, we were unable to find such a reason.

The heavy-light systems in the 't Hooft model were considered previously, in an applied context, e.g. in Ref. [10]. In this work the original light-cone 't Hooft equation was numerically solved at large values of $m_2/\sqrt{\lambda}$. As was mentioned, in the 't Hooft-like equation the limit $m_2/\sqrt{\lambda} \rightarrow \infty$ is taken *before* solving the 't Hooft equation. The appropriate limiting procedure was implemented in [11, 12]. Note that when the heavy-light meson is boosted (to put it on the light cone) the total momentum of the meson is shared between quarks proportionally to their masses. Therefore, the heavy quark will have x very close to unity while the light quark's share will be close to zero. The width of the x distribution will be proportional to $\sqrt{\lambda}/m_2 \rightarrow 0$. This fact was noted long ago [13], and was later extensively exploited in phenomenology. The light-cone wave function will have an infinitely narrow support in the limit $\sqrt{\lambda}/m_2 \rightarrow 0$ unless we rescale the variable x , so that the corresponding distribution does not shrink to a delta function but is, rather, characterized by a constant width.

The appropriate rescaling laws are as follows [11, 12]:

$$\begin{aligned} x &= 1 - \frac{\sqrt{2\lambda}}{m_2} \xi , \\ M &= m_2 + \mathcal{E} , \\ \phi(x) &= \sqrt{m_2(2\lambda)^{-1/2}} \varphi(\xi) , \end{aligned} \quad (5)$$

where m_2 is to be sent to infinity while \mathcal{E} is kept fixed (i.e. \mathcal{E} is the mass of the bound state after the subtraction of the mechanical mass of the infinitely heavy anti-quark).

Then the light-cone 't Hooft equation takes the form

$$2\mathcal{E}\varphi(\xi) = \sqrt{2\lambda}\xi\varphi(\xi) - \sqrt{2\lambda} \int_0^\infty \frac{\varphi(\tilde{\xi}) - \varphi(\xi)}{(\tilde{\xi} - \xi)^2} d\tilde{\xi} . \quad (6)$$

The boundary conditions in this equation are as follows:

$$\varphi(\xi \rightarrow 0) \rightarrow \text{const} , \quad \varphi(\xi \rightarrow \infty) \rightarrow 0 . \quad (7)$$

Our main results can be summarized as follows. We solve the heavy-light system in the laboratory frame using the Coulomb (axial) gauge. As the first step we solve the gap equation and obtain the required quark Green function. Given this quark Green function we are in position to solve the Bethe-Salpeter equation. Both the single-quark Green function (the quark condensate follows straightforwardly from the quark Green function) and the meson spectrum manifestly exhibit dynamical chiral symmetry breaking. Then we solve the same system on the light cone by integrating (numerically) the 't

Hooft-like equation. We obtain exactly the same spectrum even though the dynamical equations in both cases have very different physical meaning, and there is no gap equation on the light cone. Dynamical chiral symmetry breaking is manifest through the absence of parity doubling in the spectrum in both cases, but in the laboratory frame this chiral symmetry breaking is also clearly seen through the nonzero quark condensate in the vacuum. While all the intermediate color-nonsinglet quantities, such as the quark Green function, manifestly depend on the reference frame and on the gauge-fixing condition, the spectrum of the color-singlet system is independent of the choice of the quantization scheme, of the reference frame and of the gauge condition.

In Section II we briefly review the chiral symmetry breaking and solution of the associated gap equation in the laboratory frame. In Section III we discuss the spectral equation for the heavy-light mesons in the laboratory frame and on the light cone. Numerical solutions are presented. Section IV briefly summarizes our results and conclusions.

II. CHIRAL SYMMETRY BREAKING IN VACUUM

A. The gap equation

In the laboratory frame, the axial (Coulomb) gauge condition

$$A_1 = 0 \quad (8)$$

is convenient. The derivation of the bound state equation is carried out in two steps, see [9] for details. First one needs to obtain the quark Green's function for the massless quark. Its self-energy saturated in the large- N limit by the rainbow graphs.

To introduce necessary notation it is convenient to start, however, from the one-loop graph presented in Fig. 1.

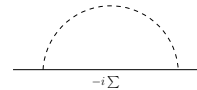


FIG. 1: Quark self-energy at one loop.

We will denote the quark self-energy by $-i\Sigma$, so that the quark Green's function is

$$\begin{aligned} G_{ij}(p_0, p) &= \int d^2x e^{ip_\mu x^\mu} \langle T \{ \psi(x) \bar{\psi}(0) \} \rangle \\ &= \frac{i}{\not{p} - m - \Sigma}, \end{aligned} \quad (9)$$

where the mass parameter m is arbitrary (real and positive) for the time being. In the $A_1 = 0$ gauge Σ depends only on the spatial component of the quark momentum p , not on p^0 . In calculating the graph of Fig. 1 we benefit from the fact that only D_{00} is non-vanishing, and perform the integral over the time component of the loop momenta using residues. In this way we arrive at

$$\begin{aligned} \Sigma(p) &= \frac{\lambda}{2} \left\{ -2\gamma^1 \left[\frac{p}{m^2 + p^2} + \frac{m^2}{2(m^2 + p^2)^{3/2}} \ln \frac{\sqrt{m^2 + p^2} + p}{\sqrt{m^2 + p^2} - p} \right] \right. \\ &\quad \left. - m \left[\frac{2}{m^2 + p^2} - \frac{p}{(m^2 + p^2)^{3/2}} \ln \frac{\sqrt{m^2 + p^2} + p}{\sqrt{m^2 + p^2} - p} \right] \right\}. \end{aligned} \quad (10)$$

Now we see that (i) The loop expansion parameter is $\lambda/(m^2 + p^2)$; it explodes at $m, p < \sqrt{\lambda}$, so that summation of the infinite series is necessary; (ii) In the $A_1 = 0$ gauge Σ depends only on the spatial component of momentum; (iii) Its general Lorentz structure is

$$\Sigma(p) = A(p) + B(p) \gamma^1, \quad (11)$$

where A and B are some real functions of p (for real p). From Eq. (9) we see that the combination we will be dealing with in the quark Green's function is

$$m + p \gamma^1 + A(p) + B(p) \gamma^1. \quad (12)$$

Usually A and B are traded for two other functions, which parametrize the quark Green's function in a more convenient way. Namely,

$$\begin{aligned} E_p &\equiv \sqrt{(m + A(p))^2 + (p + B(p))^2}, \\ m + A(p) &= E_p \cos \theta_p, \\ p + B(p) &= E_p \sin \theta_p, \end{aligned} \quad (13)$$

where for consistency one should demand E_p to be *positive* for all real p . The angle θ_p is referred to as the Bogoliubov angle, or, more commonly, the chiral angle. The exact quark Green's function now can be rewritten

as

$$G = i \frac{p^0 \gamma^0 - E_p \sin \theta_p \gamma^1 + E_p \cos \theta_p}{p_0^2 - E_p^2 + i\epsilon}. \quad (14)$$

Closed-form exact equations can be obtained for E_p and θ_p due to the fact that in the 't Hooft limit the quark self-energy is saturated by “rainbow graphs.” An example of the rainbow graph is depicted in Fig. 2. Intersections of the gluon lines and insertions of the internal quark loops are forbidden, and so are the gluon lines on the other side of the quark line. This diagrammatic structure implies an equation depicted in Fig. 3, where the bold solid line denotes the exact Green’s function (14). Algebraically

$$\Sigma(p) = \frac{i\lambda}{2\pi} \oint \frac{d^2k}{(p-k)^2} \gamma^0 G(k) \gamma^0. \quad (15)$$

It is easy to see that this equation sums up the infinite sequence of the rainbow graphs in its entirety. In Eq. (15) a principal value of the integral on the right-hand side is assumed.

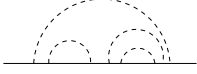


FIG. 2: An example of the rainbow graph in $\Sigma(p)$.

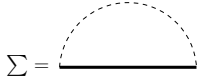


FIG. 3: Exact equation for $\Sigma(p)$ summing all rainbow graphs. The bold solid line is the exact quark propagator (14).

Using (14) and performing integration over k^0 , the time component of the loop momentum, by virtue of residues, it is not difficult to obtain

$$\Sigma(p) = \frac{\lambda}{2} \oint dk \left\{ \gamma^1 \sin \theta_k \frac{1}{(p-k)^2} + \cos \theta_k \frac{1}{(p-k)^2} \right\}, \quad (16)$$

which implies, in turn,

$$A(p) = E_p \cos \theta_p - m = \frac{\lambda}{2} \oint dk \cos \theta_k \frac{1}{(p-k)^2},$$

$$B(p) = E_p \sin \theta_p - p = \frac{\lambda}{2} \oint dk \sin \theta_k \frac{1}{(p-k)^2}. \quad (17)$$

This should be supplemented by the boundary conditions

$$\theta_p \rightarrow \begin{cases} \frac{\pi}{2} & \text{at } p \rightarrow \infty, \\ -\frac{\pi}{2} & \text{at } p \rightarrow -\infty, \end{cases} \quad (18)$$

determined by the free-quark limit. The integrals (15) – (17) contain singularity at $p = k$, so a regularization

is required. We use the principal value regularization. This set of equations, called the gap or the Schwinger–Dyson equation, was first obtained by Bars and Green [9]. Multiplying the first equation by $\sin \theta_p$ and the second by $\cos \theta_p$ and subtracting one from another one gets an integral equation for the chiral angle, namely,

$$p \cos \theta_p - m \sin \theta_p = \frac{\lambda}{2} \int dk \sin(\theta_p - \theta_k) \frac{1}{(p-k)^2}. \quad (19)$$

The latter equation, in contrast to (15) – (17), does not contain singularity at $p = k$. Assuming that the chiral angle is found in the limit $m = 0$ from

$$p \cos \theta_p = \frac{\lambda}{2} \int dk \sin(\theta_p - \theta_k) \frac{1}{(p-k)^2}, \quad (20)$$

one can get E_p from the equation

$$E_p = p \sin \theta_p + \frac{\lambda}{2} \int dk \cos(\theta_p - \theta_k) \frac{1}{(p-k)^2}. \quad (21)$$

An immediate consequence is that θ_p is an odd function of p , while $E(p)$ is even.

By solving the gap equation one obtains the chiral angle θ_p and both dressing functions $A(p)$ and $B(p)$. In the chiral limit $m = 0$ the chiral symmetry breaking part of the quark Green function is $A(p)$. Consequently a nonzero $A(p)$ signals dynamical chiral symmetry breaking in the vacuum. It is an intrinsically non-perturbative effect that cannot be obtained within the perturbation theory.

B. A wrong solution

Upon examining Eq. (20) it is not difficult to guess an analytic solution,

$$\theta_p = \frac{\pi}{2} \text{sign } p, \quad (22)$$

where $\text{sign } p$ is the sign function,

$$\text{sign } p = \vartheta(p) - \vartheta(-p).$$

The solution (22) is singular. If nevertheless we use it, then substituting (22) in Eq. (21) one obtains

$$E_p = |p| - \frac{\lambda}{|p|}. \quad (23)$$

The above results shows that the analytic solution (22) is unphysical. This is obvious from the fact that E_p becomes negative at $|p| < \sqrt{\lambda}$. This feature of the solution (23) — negativity at small $|p|$ — cannot be amended by a change of the infrared regularization. See also [14].

The unphysical solution (23) leads to the vanishing quark condensate, as will be clear from Eq. (25). We will return to the unphysical solution later, after discussing the (nonsingular) physical solution.

C. Physical solution

A solution that leads to a nonvanishing condensate has the form depicted in Fig. 4. It is smooth everywhere. At $|p| \ll \sqrt{\lambda}$ it is linear in p . Its asymptotic approach to $\pm\pi/2$ at $|p| \gg \sqrt{\lambda}$ will be discussed later.

Now, let us calculate the chiral condensate, the vacuum expectation value $\langle \bar{\psi}\psi \rangle$,

$$\langle \bar{\psi}\psi \rangle = -\text{Tr} \int \frac{d^2 p}{(2\pi)^2} G(p_0, p), \quad (24)$$

where Tr stands for both traces, with respect to color and Lorentz indices, and the quark Green function $G(p_0, p)$ is defined in Eq. (14). Taking the trace and performing the p_0 integration we arrive at

$$\langle \bar{\psi}\psi \rangle = -N \int \frac{dp}{2\pi} \cos \theta_p. \quad (25)$$

For the singular solution (22) the above quark condensate vanishes since $\cos \theta_p \equiv 0$. However, for the physical smooth solution depicted in Fig. 4 the quark condensate does not vanish,

$$\langle \bar{\psi}\psi \rangle = -\frac{N}{\sqrt{6}} \sqrt{\lambda}. \quad (26)$$

Equation (25) in conjunction with (20), allow us to determine the leading preasymptotic correction in θ_p at $|p| \gg \sqrt{\lambda}$. Indeed, in this limit the right-hand side of Eq. (20) reduces to (at $p > 0$)

$$\frac{\lambda}{2p^2} \int dk \sin \left(\frac{\pi}{2} - \theta_k \right) = \frac{\lambda}{2p^2} \int dk \cos \theta_k, \quad (27)$$

while the left-hand side

$$p \sin \left(\frac{\pi}{2} - \theta_p \right) \rightarrow p \left(\frac{\pi}{2} - \theta_p \right). \quad (28)$$

This implies, in turn, that

$$\theta_p = \frac{\pi}{2} \text{sign } p - \frac{\pi}{\sqrt{6}} \left(\frac{\sqrt{\lambda}}{p} \right)^3 + \dots, \quad |p| \gg \sqrt{\lambda}. \quad (29)$$

At the same time, from Eq. (21) we deduce that there is no p^{-3} correction in $E/|p|$, the leading correction is of order of λ^3/p^6 .

D. Numerical solution of the gap equation and an alternative scheme of regularization

The gauge choice (8) for the model (1) ensures the existence of only one non-trivial component of the gluon propagator:

$$\begin{aligned} D_{01}^{ab}(x_0 - y_0, x - y) &= D_{11}^{ab}(x_0 - y_0, x - y) = 0, \\ D_{00}^{ab}(x_0 - y_0, x - y) &= -\frac{i}{2} \delta^{ab} |x - y| \delta(x_0 - y_0). \end{aligned} \quad (30)$$

$D_{00}^{ab}(x_0 - y_0, x - y)$ corresponds to an instantaneous linear confining potential. All loop integrals calculated with a linear potential diverge in the infrared region, hence one has to introduce an infrared regularization. This can be done in a number of ways. In previous sections we used a principal value regularization.

Here we apply an alternative regularization, which suppresses the small momenta of the linear potential by introducing a cutoff parameter into the propagator in the momentum representation. We define propagator in momentum representation as

$$D_{00}^{ab}(x_0 - y_0, p) = i \frac{\delta^{ab} \delta(x_0 - y_0)}{p^2 + \mu_{IR}^2}. \quad (31)$$

Then in the final answer for the color-singlet quantities the infrared limit $\mu_{IR} \rightarrow 0$ must be taken.

In the regularization scheme defined by (31) the expression for the self-energy operator (15) turns into

$$\begin{aligned} \Sigma(p) &= \frac{\lambda}{2} \int dk \left[\gamma^1 \sin \theta_k \frac{1}{(p - k)^2 + \mu_{IR}^2} \right. \\ &\quad \left. + \cos \theta_k \frac{1}{(p - k)^2 + \mu_{IR}^2} \right]. \end{aligned} \quad (32)$$

Using the representation of the delta-function

$$\delta(x) = \lim_{\mu_{IR} \rightarrow 0} \frac{1}{\pi} \frac{\mu_{IR}}{x^2 + \mu_{IR}^2}, \quad (33)$$

it is easy to see that the self-energy defined in (32) diverges at $\mu_{IR} \rightarrow 0$ as:

$$\lim_{\mu_{IR} \rightarrow 0} \Sigma(p) = \frac{\lambda\pi}{2\mu_{IR}} \sin \theta_p \gamma^1 + \frac{\lambda\pi}{2\mu_{IR}} \cos \theta_p + \text{a finite part}. \quad (34)$$

The self-energy operator defined in (15) via the principal value regularization is always finite. This is also true for the energy of a single quark which, being regularized through (31) takes the form

$$E_p = p \sin \theta_p + \frac{\lambda}{2} \int dk \cos(\theta_p - \theta_k) \frac{1}{(p - k)^2 + \mu_{IR}^2}. \quad (35)$$

E_p diverges at $\mu_{IR} \rightarrow 0$ as

$$\lim_{\mu_{IR} \rightarrow 0} E_p = \frac{\lambda\pi}{2\mu_{IR}} + \text{finite terms}, \quad (36)$$

while with the principal value regularization it is always finite. For any other color-nonsinglet quantity one has the same situation.

This circumstance reflects the confining properties of the 't Hooft model. Confinement means that only observable color-singlet quantities have finite well-defined values, that should not depend on the infrared regularization scheme. The color-nonsinglet quantities are not observable and manifestly depend on the regularization choice. Our present regularization is convenient in the

sense that it explicitly removes all color-nonsinglet objects from the physical Hilbert space since they are all infrared divergent. At the same time this infrared divergence exactly cancels in all color-singlet observable quantities, such as the meson spectrum, the chiral angle and the quark condensate. The color-singlet quantities are finite and do not depend on the choice of the regularization.

In the following we show that the infrared divergences exactly cancel in the gap equation, written in the form

$$A(p) \sin \theta_p - [B(p) + p] \cos \theta_p = 0, \quad (37)$$

where $A(p)$ and $B(p)$ in the regularization scheme (31) are

$$\begin{aligned} A(p) &= \frac{\lambda}{2} \int dk \frac{\cos \theta_k}{(p-k)^2 + \mu_{IR}^2}, \\ B(p) &= \frac{\lambda}{2} \int dk \frac{\sin \theta_k}{(p-k)^2 + \mu_{IR}^2}. \end{aligned} \quad (38)$$

Using the representation of the delta function (33) we obtain at $\mu_{IR} \rightarrow 0$:

$$\begin{aligned} A(p) &= \frac{\lambda\pi}{2\mu_{IR}} \cos \theta_p + A_{\text{finite}}(p), \\ B(p) &= \frac{\lambda\pi}{2\mu_{IR}} \sin \theta_p + B_{\text{finite}}(p). \end{aligned} \quad (39)$$

Note that in (37) all divergences exactly cancel and

$$\tan \theta_p = \frac{B(p) + p}{A(p)} = \frac{B_{\text{finite}}(p) + p}{A_{\text{finite}}(p)}. \quad (40)$$

Equation (37) can be solved at exceedingly small but finite values of μ_{IR} ; then extrapolation to the limit $\mu_{IR} \rightarrow 0$ must be performed. The equation is solved recurrently with a special care for the numerical integration in the vicinity of $p = k$. The resulting chiral angle is consistent with previous studies [6, 7] and is presented in Fig. 4.

III. THE HEAVY-LIGHT MESONS

A. Equation for the heavy-light mesons

The Bethe-Salpeter equation for the heavy-light mesons in the laboratory frame follows from [9] in a straightforward manner, by taking the limit $m_2 \rightarrow \infty$ in the coupled equations of [9], which untangles them. The corresponding Bethe-Salpeter equation was obtained e.g. in Refs. [14, 15]; an alternative derivation can be found in the text [16]. It has the form

$$\begin{aligned} \mathcal{E}\phi(p) &= p \sin \theta_p \phi(p) \\ &- \lambda \int \frac{dk}{(p-k)^2} \left[\cos \frac{\theta_p - \theta_k}{2} \phi(k) \right. \\ &- \left. \left(\cos \frac{\theta_p - \theta_k}{2} \right)^2 \phi(p) \right]. \end{aligned} \quad (41)$$

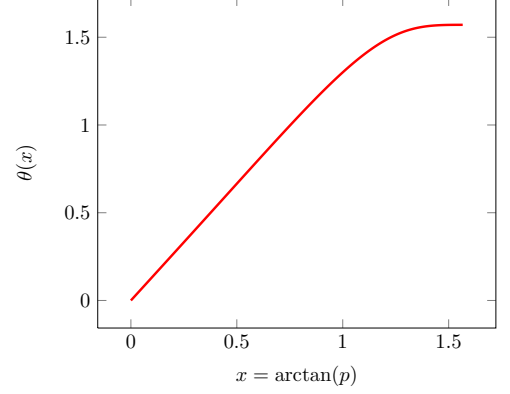


FIG. 4: Numerical solution of the gap equation for the Bogolyubov angle $\theta(p)$, p is in units of $\sqrt{\lambda}$. Where x comes from the change of variable $p = \tan(x)$.

It is not difficult to derive the boundary conditions on $\phi(p)$ and some properties of the wave function:

(i) it can be taken real, nonsingular, and either symmetric or antisymmetric under $p \rightarrow -p$,

$$\phi(-p) = \pm \phi(p);$$

and

(ii) at large $|p|$

$$\phi(p) \sim \begin{cases} \frac{1}{|p|^3} & \text{symmetric levels,} \\ \frac{1}{p^4} & \text{antisymmetric levels} \end{cases}. \quad (42)$$

This asymptotic behavior is necessary to guarantee the cancellation of the leading (at large p) term on the right-hand side of Eq. (41).

Knowing the numerical solution for the chiral angle θ_p , we are able to solve equation (41). For the numerical solution of equation (41) it is convenient to use the regularization (31). Equation (41) then takes the form

$$\begin{aligned} \mathcal{E}\phi(p) &= p \sin \theta_p \phi(p) \\ &- \lambda \int \frac{dk}{(p-k)^2 + \mu_{IR}^2} \left[\cos \frac{\theta_p - \theta_k}{2} \phi(k) \right. \\ &- \left. \left(\cos \frac{\theta_p - \theta_k}{2} \right)^2 \phi(p) \right]. \end{aligned} \quad (43)$$

Considering (43) at $\mu_{IR} \rightarrow 0$ one can see that all infrared divergences cancel each other

$$\mathcal{E}\phi(p) = p \sin \theta_p \phi(p) - \frac{\lambda\pi}{\mu_{IR}} \phi(p) + \frac{\lambda\pi}{\mu_{IR}} \phi(p) + \text{a finite part}. \quad (44)$$

We solve Eq. (43) variationally by expanding the unknown wave function in the basis

$$\phi(p) = \sum_{i=1}^N C_i \chi_i(p). \quad (45)$$

For the symmetric levels, we choose a basis in the form

$$\chi_i(p) = \exp(-\alpha_i p^2)$$

while for antisymmetric

$$\chi_i(p) = p \exp(-\alpha_i p^2).$$

A relatively small number of gaussians is required for a sufficiently accurate expansion. Given the above basis, Eq. (43) transforms into a system of linear equations

$$\begin{aligned} \mathcal{E} \sum_{i=1}^N C_i \chi_i(p) &= p \sin \theta_p \sum_{i=1}^N C_i \chi_i(p) \\ &- \lambda \int \frac{dk}{(p-k)^2 + \mu_{IR}^2} \left[\cos \frac{\theta_p - \theta_k}{2} \sum_{i=1}^N C_i \chi_i(k) - \left(\cos \frac{\theta_p - \theta_k}{2} \right)^2 \sum_{i=1}^N C_i \chi_i(p) \right]. \end{aligned} \quad (46)$$

Multiplying (46) by $\chi_j(p)$, we obtain the generalized eigenvalue problem:

$$\mathcal{E} D \vec{C}_n = (A + B) \vec{C}_n,$$

where

$$D_{ij} = \int dp \chi_i(p) \chi_j(p),$$

$$A_{ij} = \int dp p \sin \theta_p \chi_i(p) \chi_j(p), \quad (47)$$

$$\begin{aligned} B_{ij} &= \int dp \int dk \left[\cos \frac{\theta_p - \theta_k}{2} \chi_i(k) \chi_j(p) \right. \\ &\quad \left. - \left(\cos \frac{\theta_p - \theta_k}{2} \right)^2 \chi_i(p) \chi_j(p) \right]. \end{aligned} \quad (48)$$

Energy levels obtained by solving the problem (47) are shown in Table I and in Fig. 5, the corresponding wave functions are in Fig. 6 and Fig. 7. All wave functions are normalized by condition $\int dp \phi^2(p) = 1$.

TABLE I: Energy levels of the heavy-light hadrons in units of $\sqrt{\lambda}$

n	$P = -$	$P = +$
0	1.161	3.043
1	4.300	5.286
2	6.126	6.868
3	7.540	8.159
4	8.734	9.276
5	9.789	10.27
6	10.74	11.18

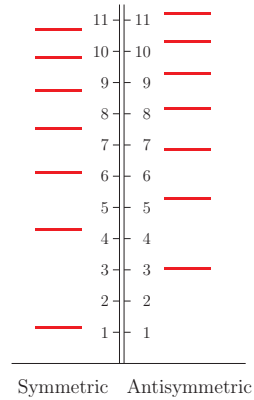


FIG. 5: Spectrum of the heavy-light mesons in units of $\sqrt{\lambda}$.

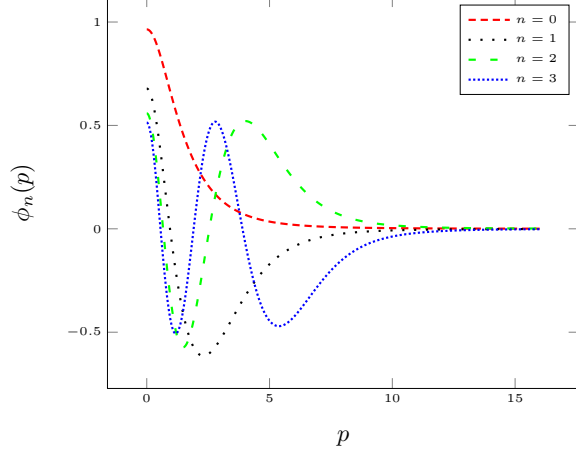


FIG. 6: Wave functions of mesons with the negative parity (i.e. with the "symmetric" relative motion wave function). The wave function $\phi_n(p)$ is in units of $\lambda^{(-1/4)}$ and momentum p is in units of $\sqrt{\lambda}$.

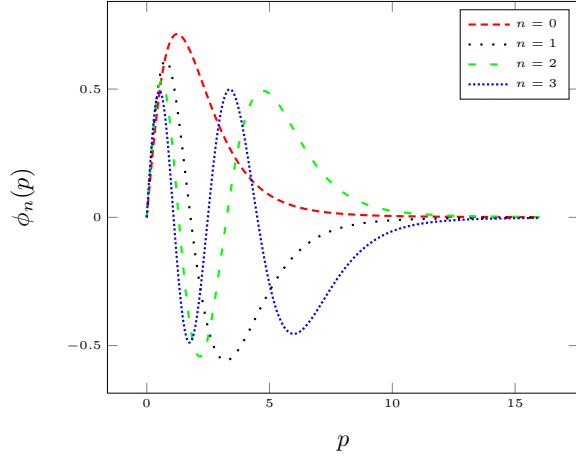


FIG. 7: Wave functions of mesons with the positive parity (i.e. with the "antisymmetric" relative motion wave function). The wave function $\phi_n(p)$ is in units of $\lambda^{(-1/4)}$ and momentum p is in units of $\sqrt{\lambda}$.

B. The heavy-light mesons on the light cone

Now we deal with the 't Hooft-like equation (6). In order to solve it numerically we split the integral into

two parts

$$2\mathcal{E}_m\varphi_m(\xi) = \sqrt{2\lambda}\xi\varphi_m(\xi) - \sqrt{2\lambda}\lim_{\epsilon\rightarrow 0}\left(\int_0^{\xi-\epsilon}\frac{\varphi_m(\tilde{\xi})-\varphi_m(\xi)}{(\tilde{\xi}-\xi)^2}d\tilde{\xi} + \int_{\xi+\epsilon}^\infty\frac{\varphi_m(\tilde{\xi})-\varphi_m(\xi)}{(\tilde{\xi}-\xi)^2}d\tilde{\xi}\right). \quad (49)$$

Alternatively the 't Hooft-like equation can be solved with definition (31). Then it takes form:

$$2\mathcal{E}_m\varphi_m(\xi) = \sqrt{2\lambda}\xi\varphi_m(\xi) - \sqrt{2\lambda}\int_0^\infty\frac{\varphi_m(\tilde{\xi})-\varphi_m(\xi)}{(\tilde{\xi}-\xi)^2+\mu_{IR}^2}d\tilde{\xi}, \quad (50)$$

where $\mu_{IR} \rightarrow 0$ is assumed.

Both equations (49) and (50) were solved numerically in much the same way as Eq. (43). The results in both cases (49) and (50) coincide. The spectrum is identical to that following from the laboratory-frame equation (43), see Fig. 5. The light-cone wave functions are normalized by the condition $\int d\xi\varphi_m^2(\xi) = 1$ and presented in Fig. 8.

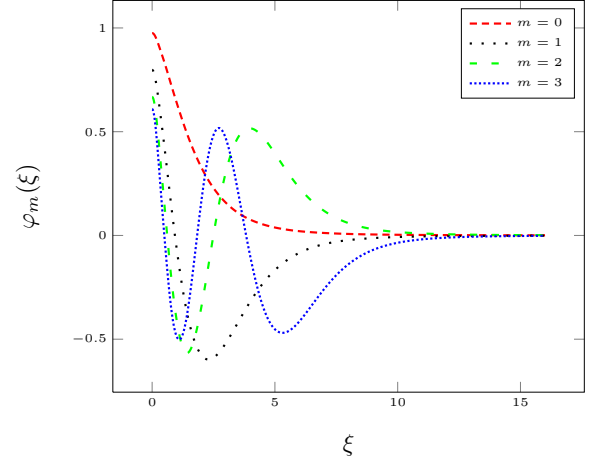


FIG. 8: Wave functions of mesons obtained from the 't Hooft-like equation. Even m represent the negative parity mesons and odd m correspond to the positive parity mesons. Both the wave functions $\varphi_m(\xi)$ and the variable ξ are dimensionless.

C. Equation (41) with the unphysical chiral angle vs. the 't Hooft-like equation

People are used to the fact that the chiral condensate cannot be directly captured if one works on the light cone. At the same time, the chiral symmetry breaking is seen

indirectly, through the absence of the parity degeneracy in the spectrum of physical mesons. The situation in our laboratory-frame construction is totally different. The nonsingular solution for θ_p , see Section IID, immediately produces $\langle \bar{\psi}\psi \rangle \neq 0$, see Eq. (25). As a result, naturally, all P -odd states split from P -even.

The singular solution (22) would lead to $\langle \bar{\psi}\psi \rangle = 0$. If, using (22), we could obtain a consistent laboratory-frame Bethe–Salpeter equation, with a proper Foldy–Wouthuysen transformation, it should have produced a parity degenerate meson spectrum, in full accord with general theorems. However, (22) implies (23), which obviously precludes the use of (22) in the Bethe–Salpeter equation because of negativity of the solution (23) at small $|p|$.

Physically it means that the chiral symmetry is *a priori* broken in the 't Hooft model. Trying to restore it by brute force insisting on the chirally symmetric vacuum, we see that the bound state equation for hadrons in the rest frame is not defined, and no consistent solutions for hadronic spectrum exists.

Nevertheless, let us perform this incorrect and illegitimate operation, and see what happens. Below we examine a strange construct, namely, Eq. (41) with the *singular* (unphysical) chiral angle, i.e. we replace $\theta_{p,k}$ in Eq. (41) by (22). This is no longer a legitimate laboratory-frame Bethe–Salpeter equation. But it has a miraculous feature.

For positive values of p we get

$$\mathcal{E}\phi(p) = p\phi(p) - \lambda \int_0^\infty \frac{dk}{(p-k)^2} [\phi(k) - \phi(p)]. \quad (51)$$

Next, we introduce dimensionless variables (marked by tildes)

$$p = \sqrt{\lambda} \tilde{p}, \quad k = \sqrt{\lambda} \tilde{k}. \quad (52)$$

The wave functions are to be understood now as functions depending on \tilde{p} , \tilde{k} rather than p , k , although we will keep using the same notation ϕ . Then, in terms of these dimensionless variables, Eq. (51) takes the form

$$\mathcal{E}\phi(p) = \sqrt{\lambda} \tilde{p} \phi(\tilde{p}) - \sqrt{\lambda} \int_0^\infty \frac{d\tilde{k}}{(\tilde{p} - \tilde{k})^2} [\phi(\tilde{k}) - \phi(\tilde{p})]. \quad (53)$$

Compare it with Eq. (49) or (50). We observe, with surprise, that Eq. (53) is identical to (49), up to a renaming of the integration variables and rescaling

$$\mathcal{E} \rightarrow \sqrt{2} \mathcal{E}_m. \quad (54)$$

Thus, the laboratory frame Bethe–Salpeter equation with the *wrong* chiral angle and the boundary conditions inappropriate for the laboratory frame equation [18] reproduces the spectrum of the (*correct*) 't Hooft-like light-cone equation up to an overall energy scale which is off

by a factor of $1/\sqrt{2}$. In particular, the ratios of the energy levels following from (51) are correct. The physical reason for this coincidence remains puzzling.

IV. CONCLUSIONS

We studied the heavy-light mesons in $(1+1)$ -dimensional QCD in the 't Hooft limit, with the emphasis on the impact of the chiral symmetry breaking both on the spectrum and wave functions. To this end we compared two alternative quantization schemes: laboratory frame Bethe–Salpeter equation with a nontrivial chiral angle and the light-cone 't Hooft-like equation which has no direct information on the chiral condensate in the vacuum. Two distinct limiting procedures leading to these two respective equations are not *a priori* interchangeable.

First, we solved the system in the laboratory frame using the Coulomb (axial) gauge. The solution proceeds via two steps. One begins from the solution of the gap equation and obtains a single-quark Green's function as well as the quark condensate in the vacuum. Chiral symmetry is manifestly dynamically broken in the vacuum. Then one solves the Bethe–Salpeter equation determining the odd and even wave functions and the spectrum. Chiral symmetry is broken in the spectrum too. The spectral results are independent on the gauge choice and on an infrared regularization scheme.

Second, we solved the same system on the light cone. In this case there is no analog of the gap equation, and vacuum is trivial. Nevertheless, the chiral symmetry is broken in the observable spectrum. Needless to say, all wave functions are totally different (they depend on variable which have very different meanings in these two schemes). While dynamical equations on the light cone and in the laboratory frame (with the Coulomb gauge) look very different, the results for the spectra are the same. We demonstrated this numerically; the question of explicitly finding an appropriate unitary transformation between both schemes remains open.

A curious fact was observed *en route*. The laboratory frame equation for $\phi(p)$ becomes identical to the 't Hooft-like equation for $\varphi(\xi)$ (see Eq. (6)) upon substitution into the laboratory-frame equation a singular (nonphysical) solution for the chiral angle with simultaneous rescaling of the overall energy scale.

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 - [16] M. Shifman, *Advanced Topics in Quantum Field Theory*, (Cambridge University Press, 2012), Section 41.
 - [17] Generally speaking, in two dimensions any continuous (e.g. chiral) symmetry cannot be spontaneously broken (which is known as the Mermin-Wagner-Coleman theorem). This is because massless Goldstone bosons would bring a long-range infrared divergence for $d = 2$. However, at $N_c = \infty$ self-interaction of Goldstone bosons vanishes (they do not interact also with all other mesons) and, consequently, at $N_c = \infty$ the chiral symmetry can and is indeed spontaneously broken in the 't Hooft model.
 - [18] The laboratory frame Bethe–Salpeter equation requires vanishing of the odd wave functions at $p = 0$, while the wave functions of the 't Hooft-like equation do not vanish at $\xi = 0$.